Journal of Mathematics and System Science 2 (2012) 179-184

# The Minimum Number of Points Taking Part in *k*-Sets in Sets of Unaligned Points

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Received: November 4, 2011 / Accepted: February 15, 2012 / Published: March 25, 2012.

**Abstract:** The study of *k*- sets is a very relevant topic in the research area of computational geometry. The study of the maximum and minimum number of *k*-sets in sets of points of the plane in general position, specifically, has been developed at great length in the literature. With respect to the maximum number of *k*-sets, lower bounds for this maximum have been provided by Erdõs et al., Edelsbrunner and Welzl, and later by Toth. Dey also stated an upper bound for this maximum number of *k*-sets. With respect to the minimum number of *k*-set, this has been stated by Erdos el al. and, independently, by Lovasz et al. In this paper the authors give an example of a set of *n* points in the plane in general position (no three collinear), in which the minimum number of points that can take part in, at least, a k-set is attained for every *k* with $1 \le k < n/2$. The authors also extend Erdos's result about the minimum number of points in general position which can take part in a *k*-set to a set of *n* points not necessarily in general position. That is why this work complements the classic works we have mentioned before.

**Key words:** *k*-set, convex hull, intersection of convex polygons.

## 1. Introduction

The search of upper and lower bounds on the number of halving lines or *k*-sets in a set of $n$ points located in the plane in general position is a problem widely reflected in the literature. Recall that a halving line in a set of $n$ points $\{p_1, ..., p_n\}$ is a line that joins two points of $\{p_1, ..., p_n\}$ leaving the same number of points of $\{p_1, ..., p_n\}$ in each half-plane ($n$ is an even number) and a *k*-set is a subset of $\{p_1, ..., p_n\}$ with *k* points that can be separated of the other points of $\{p_1, ..., p_n\}$ by a straight line.

With respect to the maximum number of *k*-sets, lower bounds for this maximum have been given by Erdõs et al. [1], and also independently by Edelsbrunner and Welzl [2]. They established a lower bound of the order O (nlogk) for the maximum number of *k*-sets. Later, Tóth [3] discovered a construction of a set of *n* points with O ($n2^{\sqrt{\log k}}$) *k*-sets for every *n* and *k* < *n*/2. Attending to upper bounds of this maximum number of *k*-sets, Dey [4] stated an upper bound of the order O ($n\,k^{\frac{1}{3}}$). Nowadays, this is the best upper bound for this number.

With respect to the minimum number of halving lines and *k*-sets, it is known that the minimum number of halving lines is $\frac{n}{2}$ [5] and the minimum number of *k*-sets is $2k+1$ [1, 6] (the authors refer to the latter fact as "Result 2" throughout the paper).

The problem of establishing the minimum number of points that can intervene in at least one *k*-sets of a given set of $n$ points was also posed by Erdõs et al. [1]. They proved that this minimum is also $2k+1$ (hereafter "Result 1"), and gave an example where this minimum is attained: $2k+1$ points are the

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vertices of a regular polygon, and the remaining points lie close enough to the centre of the polygon (this example also attains the minimum number of *k*-sets).

In this paper the authors present an example of a set of $n$ points in the plane where the minimum of $2k+1$ points taking part in a *k*-set is attained for every $k<\frac{n}{2}$ (Subsection 2.1). Furthermore, the authors prove that a similar example to the presented in Subsection 2.1 cannot be found for the minimum number of *k*-sets (Section 3). So the authors conclude that the only arrangement of points with the minimum number of *k*-sets ($2k+1$) is that described by Erdõs et al. [1] and Lovasz et al. [6].

The authors also generalize Result 1 to sets of points that are not necessarily in general position, but do not consist of a set of points on a line (Subsection 2.2).

Throughout the paper *k* and $n$ are positive integers, the following definitions also apply:

**Definition 1:** Consider a set $A$ of points in the plane and the convex hulls of all possible subsets of $A$ with $t$ points. The authors define $C_{A,t}$ as the intersection of these convex hulls.

**Remark:** The following properties for $C_{A,t}$ hold [7]:

(1) If the points of $A$ are in general position, then $C_{A,t}$ does not consist only of a segment;

(2) If $t<\frac{|A|}{2}+1$, then $C_{A,t}$ is the empty set, where $|A|$ is the cardinal of $A$;

(3) If the points of $A$ are not collinear, then $C_{A,\frac{|A|}{2}+1}\subset\{p\}$ for some point *p*.

**Definition 2:** Consider a set $A$ of points in the plane, two points $p,q\in A$ and the convex hulls of all possible subsets of $A$ with $t$ points such that $p$ and/or $q$ belongs to the subset. The authors define $C_{A,t}^{p,q}$ as the intersection of these convex hulls.

## 2. Minimum Number of Points Taking Part in *k*-Sets of *A*

### *2.1 Example for a Set of n Points and Every* $k<\frac{n}{2}$

In order to give the example of a set of $n$ points, with even $n$, with the minimum number of points taking part in at least one *k*-set for every $k<\frac{n}{2}$, the authors shall need some previous results. Throughout this Subsection it is assumed that the points of every set are in general position:

**Proposition 1:** Let $A$ be a set of $n$ points. The points of $A$ included in $C_{A,n-k}$ cannot belong to any *k*-set.

**Proof:** If one of these points belonged to a $k-set$, then a straight line would separate it from $n-k$ points of $A$. Therefore, this point would not be included in at least one convex hull of $n-k$ points and could not belong to $C_{A,n-k}$, a contradiction.

**Remark:** Conversely, the points of $A$ that are not included in $C_{A,n-k}$ belong to at least a *k*-set. Consequently the authors wish to find an example of a set $A$ of $n$ points such that $n-(2k+1)$ points belong to $C_{A,n-k}$ for every $k$ in the range $1\le k<\frac{n}{2}$.

**Lemma 1:** Let *U* and *V* be the sets $U=\{p_1,...,p_t\}$, $V=\{p_1,...,p_t,p_{t+1},p_{t+2}\}$, where $t$ is an odd number. If the points $p_{t+1}$ and $p_{t+2}$ belong to $C_{U,\left[\frac{t}{2}\right]+2}$, then these points also belong to $C_{V,\left[\frac{t+2}{2}\right]+2}$. Furthermore, $C_{V,\left[\frac{t+2}{2}\right]+2}$ has a non empty interior set ([ ] stands for the floor).

**Proof:** Consider a set of $\left[\frac{t+2}{2}\right]+2=\left[\frac{t}{2}\right]+3$ points of *V*. If these points do not include both $p_{t+1}$ and $p_{t+2}$, then they will contain at least $\left[\frac{t}{2}\right]+2$ points of *U*. Thus, the convex hull of the $\left[\frac{t}{2}\right]+3$ points considered must contain the convex hull of $\left[\frac{t}{2}\right]+2$ points of *U*.

Consequently, the first convex hull contains the segment joining $p_{t+1}$ and $p_{t+2}$ by the hypothesis of the lemma.

Now, if the set of $\left[\frac{t+2}{2}\right]+2$ points of $V$ considered contains both $p_{t+1}$ and $p_{t+2}$, then the segment joining $p_{t+1}$ and $p_{t+2}$ is included in the convex hull. This segment is therefore in $C_{V,\left[\frac{t+2}{2}\right]+2}$ and consequently $C_{V,\left[\frac{t+2}{2}\right]+2}$ is not a finite set. But the set $C_{V,\left[\frac{t+2}{2}\right]+2}$ does not consist only of this segment, because the points are in general position. Hence, $C_{V,\left[\frac{t+2}{2}\right]+2}$ has non empty interior set.

**Lemma 2:** Consider a set of $n$ points $A=\{p_1,\ldots,p_n\}$ and its subset $B=\{p_1,\ldots,p_{2k+1}\}$. If $C_{B,\left[\frac{2k+1}{2}\right]+2}$ contains the $n-(2k+1)$ points of $A-B$, then $C_{A,n-k}$ also contains these $n-(2k+1)$ points of $A$.

**Proof:** Consider a subset of $n-k$ points taken from $A$. If this subset does not contain all of the last $n-(2k+1)$ points of $A$ ($p_{2k+2},\ldots,p_n$), then there are at least $k+2=\left[\frac{2k+1}{2}\right]+2$ points in subset $B$, so their convex hull contains the last $n-(2k+1)$ points of $A$ by assumption, then $p_{2k+2},\ldots,p_n$ are in $C_{A,n-k}$.

Let us next describe the example satisfying the required conditions:

**Example 1**

Let $A=\{p_1,\ldots,p_n\}$ be a set of $n$ points ($n$ is an even number) defined in the following way: $p_1$, $p_2$, $p_3$ are not in a line, and for $k=1,\ldots,\frac{n-4}{2}$, $p_{2k+2}$, $p_{2k+3}$ are in $C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$ in such way that $p_1,\ldots,p_{2k+3}$ are in general position (this can always be done, since $C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$ has non empty interior set by Lemma 1). Finally, $p_n$ is located in $C_{\{p_1,\ldots,p_{n-1}\},\left[\frac{n-1}{2}\right]+2}$ (Fig. 1).

This configuration of points satisfies the condition that for every $k=1,\ldots,\frac{n-4}{2}$, $p_{2k+2},\ldots,p_n$ belong to

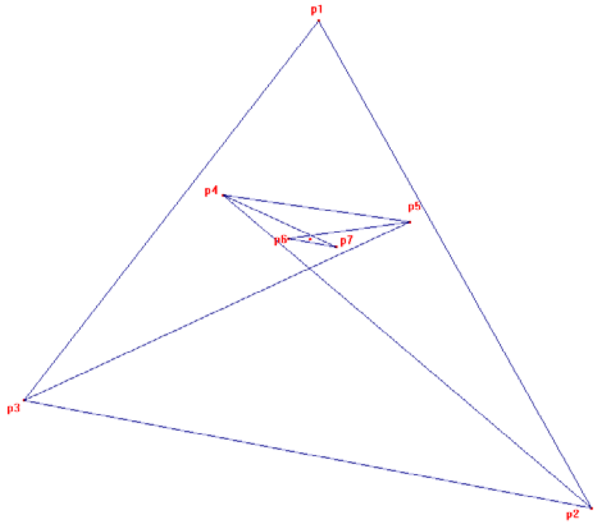


**Fig. 1 The set of the example for $n=8$.**

$C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$. The authors already know that $p_{2k+2}, p_{2k+3}$ belong to $C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$. Hence, to prove the assertion it is enough to see that $C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2} \supset C_{\{p_1,\ldots,p_{2t+1}\},\left[\frac{2t+1}{2}\right]+2}$ for $t>k$. This relation will be true for all $t>k$ if the authors see it for $t=k+1$. The following inclusion is obvious:

$$C_{\{p_1,\ldots,p_{2k+3}\},\left[\frac{2k+3}{2}\right]+2} \subset C^{p_{2k+2},\,p_{2k+3}}_{\{p_1,\ldots,p_{2k+3}\},\left[\frac{2k+3}{2}\right]+2}.$$

On the other hand, consider a selection of $\left[\frac{2k+3}{2}\right]+2$ points from the sequence $p_1,\ldots,p_{2k+3}$. Assuming that $p_{2k+2}$ and/or $p_{2k+3}$ are included, this selection contains at most $\left[\frac{2k+1}{2}\right]+2$ points from the sequence $p_1,\ldots,p_{2k+1}$. Therefore, the convex hull of the $\left[\frac{2k+3}{2}\right]+2$ points is contained within a convex hull of $\left[\frac{2k+1}{2}\right]+2$ points from $p_1,\ldots,p_{2k+1}$. This result follows from the fact that $p_{2k+2}$ and $p_{2k+3}$ are in every convex hull of $\left[\frac{2k+1}{2}\right]+2$ points taken from the sequence $p_1,\ldots,p_{2k+1}$.

Thus $C^{p_{2k+2},\,p_{2k+3}}_{\{p_1,\ldots,p_{2k+3}\},\left[\frac{2k+3}{2}\right]+2} \subset C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$

This completes the desired inclusion.

For $k=\frac{n-2}{2}$, it is also true that the point $p_{2k+2}=p_n$ is in $C_{\{p_1,\ldots,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}=C_{\{p_1,\ldots,p_{n-1}\},\left[\frac{n-1}{2}\right]+2}$, according to the construction of $A$.

Thus, according to Lemma 2 there are $n-(2k+1)$ points in $C_{A,n-k}$ for $k=1,...,\frac{n}{2}-1$. Therefore, by Proposition 1 this is an example of a set of *n* points that attains the minimum of $2k+1$ points taking part in *k*-sets for every $k=1,...,\frac{n}{2}-1$.

**Remarks:**

(1) For odd $n$, the previous example can be modified to obtain an example of a set of $n$ points with the minimum number of $2k+1$ points belonging to at least one *k*-set for every $k<\frac{n}{2}$. The authors just avoid placing the last point in the last intersection.

(2) As Fig. 1 shows, $C_{\{p_1,...,p_{2k+1}\},\left[\frac{2k+1}{2}\right]+2}$ is a triangle such that $p_{2k}$, $p_{2k+1}$ are two of its vertices.

(3) It is not possible to obtain a similar example where the minimum number of *k*-sets in a set of $n$ points is attained for every $k<\frac{n}{2}$, because this example would contradict the lower bound on the number of $\leq k$ -sets given by Lovasz [6] that is $3\binom{k+1}{2}$. As a matter of fact, it is easy to see that the number of *k*-sets in the present example is $4k-1$ for every $k<\frac{n}{2}$, $2k+1$ being the minimum number of *k*-sets.

*2.2 Case of Points That Are Not in General Position*

This Subsection generalises Result 1 by proving that for every $k<\left[\frac{n}{2}\right]$ and every set of $n$ points, the minimum number of points taking part in *k*-sets is $2k+1$, provided that the *n* points are not collinear. A previous lemma is given:

**Lemma 3:** For a set $A=\{p_1,...,p_n\}$, if $C_{A,n-k}$ contains $l$ points of $A$, say $p_1$, …, $p_l$, then these points must be located in $C_{\{p_{l+1},...,p_n\},n-k-(l-1)}$ ($l<n-k+1$).

**Proof:** If there is some point of $p_1$, …, $p_l$ that is not located in the proposed intersection, then there exists a convex hull $C$ of $n-k-(l-1)$ points of $p_{l+1},...,p_n$ that does not contain every point of $p_1$, …, $p_l$. But if such is the case, at least one point of $p_1$, …, $p_l$, for example $p_1$ is located at a vertex along the boundary of the convex hull of $p_1$, …, $p_l$ and the $n-k-(l-1)$ points aforementioned. This implies that the convex hull of the following points of $A$, $p_2$, …, $p_l$ and the $n-k-(l-1)$ points defining $C$, does not contain $p_1$, a contradiction because $p_1\in C_{A,n-k}$.

Hence $p_1$, …, $p_l$ are in $C_{\{p_{l+1},...,p_n\},n-k-(l-1)}$.

**Remark:** If $l=n-2k+1$, then $n-k-(l-1)=k$ with $k<\frac{|\{p_{l+1},...,p_n\}|}{2}+1$, so the set $C_{\{p_{l+1},...,p_n\},n-k-(l-1)}$ is empty. In this case $p_1$, …, $p_l$ cannot be included in the set. Consequently the maximum number of points of $A$ that can be located in $C_{A,n-k}$ is $n-2k$. This maximum is always attained if the $n$ points of $A$ are arranged in a line.

Next, it is can be seen that this is the only case in which the maximum number of points in $C_{A,n-k}$ is attained.

**Proposition 2:** If the maximum of $n-2k$ points of $A$ inside $C_{A,n-k}$ is attained, then the $n$ points of $A$ are in a straight line ($k<\left[\frac{n}{2}\right]$).

**Proof:** If there are $n-2k$ points of $A=\{p_1,...,p_n\}$, say $p_1$, …, $p_{n-2k}$, included in $C_{A,n-k}$, then by Lemma 3 the authors find that $p_1$, …, $p_{n-2k}$ must belong to $C_{\{p_{n-2k+1},...,p_n\},k+1}$.

If $p_{n-2k+1}$, …, $p_n$ are not collinear, then they have $C_{\{p_{n-2k+1},...,p_n\},k+1}\subset\{p\}$. (since $k+1=\frac{|\{p_{n-2k+1},...,p_n\}|}{2}+1$).

Hence, because $p_1$, …, $p_{n-2k}$ are in $C_{\{p_{n-2k+1},...,p_n\},k+1}$, the authors necessarily have that $n-2k=1$ and thus $k=\frac{n-1}{2}=\left[\frac{n}{2}\right]$, in contradiction with the condition $k<\left[\frac{n}{2}\right]$. Consequently, $p_{n-2k+1}$, …, $p_n$ are in a line, and $C_{\{p_{n-2k+1},...,p_n\},k+1}$ is included in this line. This implies that $p_1$, …, $p_{n-2k}$ are also in the line, so all $n$ points of $A$ are aligned.

Thus, if $k<\left[\frac{n}{2}\right]$ and the $n$ points of a set $A$ are not in the same line, then the maximum number of points of $A$ that can be included in $C_{A,n-k}$ is $n-(2k+1)$. This yields the statement that the authors wanted to prove:

**Corollary:** If $k<\left[\frac{n}{2}\right]$ and the $n$ points of a set $A$ are not collinear, then the minimum number of points of $A$ taking part in some *k*-set is $2k+1$.

## 3. Minimum Number of *k*-Sets

Remark 2 of Subsection 2.1 states that it is impossible to find an example similar to Example 1 for the minimum number of *k*-sets. This section proves that for a set of $n$ points, the minimum number of *k*-sets can be attained for at most one value of $k$. This minimum is necessarily attained in an example equivalent to the one shown in Erdõs et al. Ref. [1] and Lovasz et al. Ref. [6].

**Proposition 3:** For $k<\frac{n}{2}$, if the minimum number of $2k+1$ *k*-sets is attained in a set of $n$ points in general position $A=\{p_1,\ldots,p_n\}$, then there is a subset of $2k+1$ points of the set *A*, say $B=\{p_1,\ldots,p_{2k+1}\}$ in the boundary of the convex hull of the points of *A*. The other points are in $C_{B,\left[\frac{2k+1}{2}\right]+2}$.

**Proof:** If the minimum number of $2k+1$ *k*-sets is attained in a set *A*, then there can be only $2k+1$ points taking part in *k*-sets, because a distinct *k*-set can be attached to each point belonging to some *k*-set [1]. Therefore, the other $n-(2k+1)$ points must be in $C_{A,n-k}$ (Proposition 1). But then the number of $(\leq k)$ -sets in $A$ is $(2k+1)k$ and the number of $(\leq(k-1))$ -sets is $(2k+1)k-(2k+1)=(2k+1)(k-1)$. But this is the maximum number of $(\leq(k-1))$-sets when there are just $m=2k+1$ points of the set taking part in them being $m>2(k-1)+1$. Hence, the $2k+1$ points must be in a convex configuration [4]. The other points must be in $C_{B,\left[\frac{2k+1}{2}\right]+2}$ because they don't belong to any *k*-set.

To end this section, let us show that Result 2 cannot be generalised to points not in a line in the same way as Result 1:

**Example 2**

Consider a set of eight points, seven in a line and one out of line, as shown in Fig. 2.

This set only has four 3-sets: $\{1,2,3\}$, $\{1,2,8\}$, $\{5,6,7\}$ and $\{6,7,8\}$. This number is less than $2k+1=7$.

## 4. Conclusions

This paper complements some of the results contained in Erdõs et al. Ref. [1]. One of their findings, referred to as Result 1 in this paper, was that for a set of *n* points in general position, the minimum number of points taking part in *k*-sets is 2*k*+1 if $k<\frac{n}{2}$. Erdõs et al. [1] offered an example of a set of *n* points where this minimum is attained for a single value of *k*.

One improvement offered by the presented paper is an example where the lower bound of 2*k* + 1 -sets is attained for every $k<\frac{n}{2}$. According to the notation of Ábrego et al. [8] this is an example of a set with exactly two points in the *k*-layer, for every $k$ with $1<k<\frac{n}{2}$.

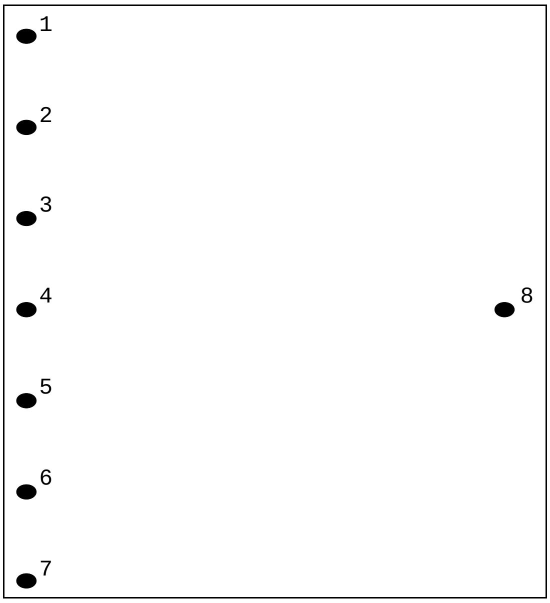


**Fig. 2 A set of points is not in general position with fewer than 2*k* + 1 *k*-sets.**

The other main improvement is the extension of Result 1 to any set of $n$ points not arranged in a line.

The authors next analysed another theorem of Erdõs et al. [1] referred to here as Result 2. This theorem states that the minimum number of $k$-sets in a set of $n$ points in general position is also $2k+1$.

The present paper proves that the example provided for Result 2 in the literature, where the minimum number of $k$-sets is attained, is essentially the only possible example.

Finally, the authors provide an example to prove that Result 2 cannot be generalised in the same way as Result 1, for any set of unaligned points.

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